

On Turan's (3,4)-problem with forbidden configurations

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Abstract

We identify three 3-graphs on five vertices each missing in all known extremal configurations for Turan's (3,4)-problem and prove Turan's conjecture for 3-graphs that are additionally known not to contain any induced copies of these 3-graphs. Our argument is based on an (apparently) new technique of “indirect interpretation” that allows us to retrieve additional structure from hypothetical counterexamples to Turan's conjecture, but in rather loose and limited sense. We also include two miscellaneous calculations in flag algebras that prove similar results about some other additional forbidden subgraphs.

1. Introduction

In the classical paper [Man07], Mantel determined the minimal number of edges a graph G with a given number of vertices must have so that every three vertices span at least one edge. In the paper [Tur41] (that essentially started off the field of extremal combinatorics), Turán generalized Mantel's result to independent sets of arbitrary size. He also asked if similar generalizations can be obtained for hypergraphs, and these questions became notoriously known ever since as one of the most difficult open problems in discrete mathematics.

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To be more specific, for a family \mathcal{H} of r -uniform hypergraphs (r -graphs in what follows) and an integer n , let $\text{ex}_{\min}(n; \mathcal{H})$ be the minimal possible number of edges in an n -vertex r -graph not containing any of $H \in \mathcal{H}$ as an induced subgraph, and let

$$\pi_{\min}(\mathcal{H}) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{\text{ex}_{\min}(n; \mathcal{H})}{\binom{n}{r}}$$

(it is well-known that this limit exists). For $\ell > r \geq 2$, let I_ℓ^r be the empty r -graph on ℓ vertices. Then we still do not know $\pi_{\min}(I_\ell^r)$ for *any* pair $\ell > r \geq 3$. More information on the history and state of the art for this and related problems can be found in the recent comprehensive survey [Kee11], and we will henceforth concentrate on the simplest case $r = 3$, $\ell = 4$. Turán’s conjecture says that $\pi_{\min}(I_4^3) = 4/9$, and it is also sometimes called *Turán’s (3,4)-problem* or *tetrahedron problem*. De Caen [Cae91], Giraud (unpublished) and Chung and Lu [CL99] proved increasingly stronger lower bounds on $\pi_{\min}(I_4^3)$, with the current record being

$$\pi_{\min}(I_4^3) \geq 0.438334$$

[Raz10, FRV12].

A prominent way to attack a difficult extremal problem is by first better understanding the nature of its (conjectured) extremal configurations and then gradually trying to solve this problem in a “neighborhood” of this set which is as “large” and “natural” as possible (we deliberately leave many terms in this sentence imprecise). In the context of Turán’s (3,4)-problem, the first part of this program can be reasonably said to have been successfully completed in the series of classical papers by Turán himself [Tur41], Brown [Bro83], Kostochka [Kos82] and Fon-der-Flaass [FdF88].

For the second part, Razborov [Raz10] proved that

$$\pi_{\min}(I_4^3, G_3) = 4/9, \tag{1}$$

where G_3 is the 3-graph on 4 vertices with 3 edges. In another paper [Raz11] of the same author, Turán (3,4)-problem was settled for two broad classes of 3-graphs resulted from the Fon-der-Flaass construction (its details will be reviewed below), each of these two classes including *all Turán-Brown-Kostochka examples*.

On the contrary, the result (1) has the obvious limitation that the additional 3-graph G_3 is missing only in Turán’s original example. In fact,

Pikhurko [Pik11] proved that his example is essentially the *only* extremal configuration for the extremal problem (1).

The main purpose of this note is to circumvent this by showing that

$$\pi_{\min}(I_4^3, H_1, H_2, H_3) = 4/9, \quad (2)$$

where H_i are 3-graphs on 5 vertices with the following sets of 3-edges:

$$\left. \begin{aligned} E(H_1) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(345)\} \\ E(H_2) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(135)(145)(235)(245)\} \\ E(H_3) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(135)(145)(235)(245)\} \end{aligned} \right\} \quad (3)$$

In words: in each of these 3-graphs, $\{1, 2, 3, 4\}$ span a clique, and the link of the remaining vertex 5 is either a perfect matching (H_1) or the complement to a perfect matching (H_2) or the complement to a single edge (H_3). Unlike G_3 , our new forbidden 3-graphs *are* missing in all Turán-Brown-Kostochka configurations (see Claim 2.2 below).

While we do not have immediate ideas how to get rid of H_1, H_2, H_3 in (2) (obvious attempts encounter the same kind of complications that in general make extremal problems for hypergraphs so painstakingly difficult), we would like to emphasize that our methods use some (apparently) novel ideas that might be of independent interest. Specifically, we are trying to utilize one of the results in [Raz11] that solves Turán’s (3,4)-problem for the class of 3-graphs resulting, via Fon-der-Flaas interpretation, from a class of oriented graphs. In the ideal world, one would have to take an arbitrary (I_4^3, H_1, H_2, H_3) -free 3-graph G and somehow create an useful oriented graph from this class to which we could apply that result. We, however, doubt very much that this goal can be easily achieved in its entirety.

Our (apparently) new observation is that our final purpose is so limited that we actually need something much more modest and relaxed than this “global” structure. Namely, we can assume that there exists a *fixed* 3-graph H that is *not* realizable via the Fon-der-Flaass construction and that has a *constant density* in a hypothetical counterexample G . The latter fact allows us to apply standard machinery from Ramsey Theory to find in G several extremely well-positioned copies of H . And then we attempt to create the Fon-der-Flaass structure on H by retrieving information encoded in

this gadget. It turns out that H_1, H_2, H_3 are precisely the only obstacles for completing this project (that, as we assumed from the beginning, can not be completed).

As we said, this method appears to be rather general and uses very little of the specifics of the Turan's (3,4)-problem. This, however, does not apply to the last (information retrieval) part that, on the contrary, seems to shed at least some light on the otherwise quite mysterious nature of the Kostochka-Fon-der-Flaass examples.

We also include two more results proved via routine calculations in Flag Algebras. The first improves upon (1) by replacing G_3 with the 3-graph having the edge set $\{(123)(124)(134)(235)\}$, and the second establishes the better bound $\pi_{\min}(I_4^3, M_2) \geq 0.4557$, where M_2 is the only I_4^3 -free 3-graph with six vertices and six edges.

2. Preliminaries

Let $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$, and, for a finite set X , let $[X]^k$ be the collection of its k -element subsets. $[[n]]^k$ is abbreviated to $[n]^k$.

In this paper we will be working with 3-graphs that will be normally denoted by the letters G, H , possibly with indices, and with oriented graphs, or *orgraphs* (directed graphs without loops, multiple or anti-parallel edges) typically denoted by Γ . In both cases, $V(\cdot)$ will be the set of vertices, and $E(\cdot)$ will be the set of 3-edges/oriented edges. For $V_0 \subseteq V(G)$, $G|_{V_0}$ is the 3-graph induced on V_0 , and likewise for $\Gamma|_{V_0}$. Two vertices u, v are *independent* in an orgraph Γ if neither $\langle u, v \rangle$ nor $\langle v, u \rangle$ is in $E(\Gamma)$.

Given a finite set \mathcal{H} of 3-graphs and an integer n , let $\text{ex}_{\min}(n; \mathcal{H})$ be the minimal possible number of edges in an n -vertex r -graph not containing any of $H \in \mathcal{H}$ as an induced subgraph, and let

$$\pi_{\min}(\mathcal{H}) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{\text{ex}_{\min}(n; \mathcal{H})}{\binom{n}{r}}$$

(it is well-known that this limit exists). Let I_4^3 be the empty 3-graph on four vertices, and let H_1, H_2, H_3 be the 3-graphs defined by their edge sets in (3).

Then our main result reads as follows:

Theorem 2.1 $\pi_{\min}(I_4^3, H_1, H_2, H_3) = 4/9$.

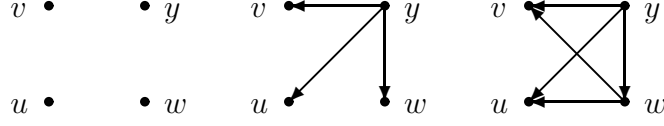


Figure 1: $\{u, v, w, y\} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$

Since, besides developing some interesting techniques, our primary motivation stems from the fact that H_1, H_2, H_3 are missing in all Kostochka configurations, we begin with a (relatively easy) clarification and verification of this fact. The reader interested only in the proof of Theorem 2.1 may safely skip this digression.

Let $\Omega = \mathbb{Z}_3 \times \mathbb{R}$, and consider the (infinite) orgraph $\Gamma_K = (\Omega, E_K)$ given by

$$E_K \stackrel{\text{def}}{=} \{ \langle (a, x), (b, y) \rangle \mid (x + y < 0 \wedge b = a + 1) \vee (x + y > 0 \wedge b = a - 1) \}.$$

Γ_K does not contain induced oriented cycles \vec{C}_4 .

For an arbitrary \vec{C}_4 -free orgraph Γ , let $FDF(\Gamma)$ be the 3-graph with $V(FDF(\Gamma)) = V(\Gamma)$ in which (uvw) is declared to span an edge if and only if $\Gamma|_{\{u,v,w\}}$ either contains an isolated (that is, of both in-degree and out-degree 0) vertex, or contains a vertex of out-degree 2 (*Fon-der-Flaass interpretation* [FdF88]). Then $FDF(\Gamma)$ is I_4^3 -free, and all known extremal configurations when the number of vertices is divisible by three are precisely of the form $FDF(\Gamma_K)|_{\mathbb{Z}_3 \times S}$ for a finite $S \subseteq \mathbb{R}$.

Claim 2.2 $FDF(\Gamma_K)$ does not contain induced copies of H_1, H_2, H_3 .

Proof. Assume the contrary, and let $\omega_i = (a_i, x_i)$ ($i \in [5]$) defines an induced embedding of some $H \in \{H_1, H_2, H_3\}$ into $FDF(\Gamma_K)$. We assume w.l.o.g. that the real numbers x_i are pairwise distinct. We treat the cases $H \in \{H_1, H_2\}$ and $H = H_3$ separately.

Case 1. $H = H_1$ OR $H = H_2$.

$\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a clique in $FDF(\Gamma_K)$. It is easy to see then that in the orgraph Γ_K these four vertices must span one of the three orgraphs on Figure 1.

If $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ are independent in Γ_K (that is, if $a_1 = a_2 = a_3 = a_4$), then ω_5 may not be independent of them, and its *link* L in $FDF(\Gamma_K)|_{\{\omega_1, \omega_2, \omega_3, \omega_4\}}$

defined as

$$L \stackrel{\text{def}}{=} \left\{ e \in [\{\omega_1, \omega_2, \omega_3, \omega_4\}]^2 \mid e \cup \{\omega_5\} \in E(FDF(\Gamma_K)) \right\}$$

will be a clique on the set

$$\{z \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \mid \langle \omega_5, z \rangle \in E(\Gamma_K)\}. \quad (4)$$

None of the graphs H_1, H_2 has this form.

Assume now that $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ span $\vec{K}_{1,3}$ in Γ_K , as shown on Figure 1, middle picture. If ω_5 is independent from $\{u, v, w\}$ then $\{u, v, w, \omega_5\}$ is another clique in $FDF(\Gamma_K)$, if ω_5 is independent from y then y is an isolated vertex in the link L , and if $\langle y, \omega_5 \rangle \in E(\Gamma_K)$ then y has degree 3 in L . None of these may happen in H_1 or H_2 so we are left with the case when $\langle \omega_5, y \rangle \in E(\Gamma_K)$ and ω_5 is also connected to the vertices u, v, w . The again L will be the clique on the same set (4) which is impossible. This completes the analysis of the second case on Figure 1.

Finally, assume that $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ span in Γ_K the third orgraph from Figure 1. By the same token as in the previous case, ω_5 may not be independent of y and $\langle y, \omega_5 \rangle \notin E(\Gamma_K)$. Thus, as before, we necessarily have $\langle \omega_5, y \rangle \in E(\Gamma_K)$.

If ω_5 is independent from u, v , then $\langle \omega_5, w \rangle \in E(\Gamma_K)$ since otherwise y again would be isolated in the link L of ω_5 . But then the configuration spanned by y, w, u, ω_5 is impossible in Γ_K . Indeed, if, say, $u = \omega_i$ then the two vertices y, w force the opposite inequalities between x_i and x_5 (recall that $\omega_j = (a_j, x_j) \in \mathbb{Z}_3 \times \mathbb{R}$).

Finally, if ω_5 is independent of w , then either $\langle \omega_5, u \rangle \in E(\Gamma_K)$ or $\langle \omega_5, v \rangle \in E(\Gamma_K)$ (otherwise y would be isolated in L), and we assume w.l.o.g. that $\langle \omega_5, u \rangle \in E(\Gamma_K)$. But now $\langle \omega_5, v \rangle \notin E(\Gamma_K)$ (as otherwise (v, u, y) would be a triangle in L) and $\langle v, \omega_5 \rangle \notin E(\Gamma_K)$ (as otherwise (v, w, y) would be an independent set in L). This contradiction completes the analysis in Case 1.

Case 2. $H = H_3$.

Since $(\omega_3\omega_4\omega_5) \notin E(FDF(\Gamma_K))$, either these three vertices span in Γ_K the oriented cycle \vec{C}_3 or, after a suitable re-numeration, $\langle \omega_3, \omega_4 \rangle \in E(\Gamma_K)$ while ω_3 and ω_5 are independent. In the first case, if ω_1 is independent of (say) ω_3 and (say) $\langle \omega_3, \omega_4 \rangle \in E(\Gamma_K)$, this implies one more missing edge $(\omega_1\omega_3\omega_4)$ in $FDF(\Gamma_K)$. In the second case, by the same token neither of ω_1, ω_2 can be independent of ω_3, ω_5 . But then at least two of the vertices $\omega_1, \omega_2, \omega_4$ must be independent, and then these two vertices along with ω_3, ω_5 may span at

most two edges in H which may not happen since $H = H_3$. Claim 2.2 is proved. ■

Along with \vec{C}_4 , the orgraph Γ_K does not contain induced copies of \bar{P}_3 (an oriented edge plus an isolated vertex). One of the two main results in [Raz11] settles Turan's (3,4)-problem for 3-graphs obtained via the Foner-Flaass interpretation from an oriented graph with the latter property. Before stating this result here, let us introduce a (tiny) bit of the flag algebra formalism [Raz07] that we will need in the next section anyway.

For two 3-graphs H, G with $|V(H)| \leq |V(G)|$ we let $p(H, G)$ denote the *density* of induced copies of H in G . We let ρ denote the 3-graph on three vertices consisting of a single edge; thus, $p(\rho, G)$ is simply the edge density of G . We have the following basic *chain rule* (see e.g. [Raz07, Lemma 2.2]):

$$p(H, \hat{G}) = \sum_{G \in \mathcal{M}_\ell} p(H, G)p(G, \hat{G}), \quad (5)$$

where $|V(H)| \leq \ell \leq |V(\hat{G})|$ and \mathcal{M}_ℓ is the set of all 3-graphs on ℓ vertices, up to an isomorphism.

Proposition 2.3 ([Raz11, Theorem 2.3]) *For any increasing sequence $\{\Gamma_n\}$ of oriented graphs without induced copies of \vec{C}_4 or \bar{P}_3 , $p(\rho, FDF(\Gamma_n)) \geq \frac{4}{9}(1 - o(1))$.*

Next, we need a multi-partite version of Ramsey's theorem that we formulate here at the level of generality sufficient for our purposes.

Proposition 2.4 ([GRS90, Theorem 5.1.5]) *For any $\ell > 0$, $n > 0$ and $r_1, \dots, r_\ell > 0$ there exists $N > 0$ such that if $|B_i| = N$ ($1 \leq i \leq \ell$) and $[B_1]^{r_1} \times \dots \times [B_\ell]^{r_\ell}$ is colored in two colors then there exist $A_i \subseteq B_i$ ($|A_i| = n$) such that $[A_1]^{r_1} \times \dots \times [A_\ell]^{r_\ell}$ is monochromatic.*

It is easy to iterate this statement to get the following.

Claim 2.5 *For any $\ell, n, r > 0$ there exists $N > 0$ such that the following holds. Let $B = B_1 \dot{\cup} \dots \dot{\cup} B_\ell$, where $|B_i| = N$, and assume that $[B]^r$ is colored in two colors. Then there exist $A_i \subseteq B_i$ ($|A_i| = n$) such that for any $E \in [A_1 \cup \dots \cup A_\ell]^r$, its color depends only on the tuple of cardinalities $\langle |E \cap A_1|, \dots, |E \cap A_\ell| \rangle$.*

Proof. For every partition $r = r_1 + \dots + r_\ell$ ($r_i \geq 0$), our original coloring of $[B]^r$ in a natural way induces a coloring of $[B_1]^{r_1} \times \dots \times [B_\ell]^{r_\ell}$. Now we simply apply recursively Proposition 2.4 to all these partitions (arbitrarily ordered). ■

When $r_1 = \dots = r_\ell = 1$, Proposition 2.4 also has a density version that, in a slightly different form, has been extensively used in areas like Additive Combinatorics, Extremal Combinatorics and Complexity Theory.

Proposition 2.6 ([GRS90, Theorem 5.1.4]) *For all $\ell > 0$, $n > 0$ and $\delta > 0$ there exists $N_0 > 0$ so that if $|B_i| = N$ ($1 \leq i \leq \ell$) with $N \geq N_0$ and $S \subseteq B_1 \times \dots \times B_\ell$ has $|S| \geq \delta N^\ell$, then there exist $A_i \subseteq B_i$ ($|A_i| = n$) such that $A_1 \times \dots \times A_\ell \subseteq S$.*

Now we have all the ingredients necessary to prove our main result.

3. Proof of Theorem 2.1

Fix an increasing sequence $\{G_m\}$ of 3-graphs not containing I_4^3, H_1, H_2, H_3 as induced subgraphs. We have to prove that $\liminf_{m \rightarrow \infty} p(\rho, G_m) \geq 4/9$. Assume the contrary, then (by restricting to a sub-sequence) we can assume w.l.o.g. that

$$p(\rho, G_m) \leq 4/9 - \epsilon \quad (6)$$

for a fixed $\epsilon > 0$ and all m .

Let us call a 3-graph G *regular* if $E(G) \supseteq E(FDF(\Gamma))$ for some orgraph Γ on the same set of vertices $V(G)$ without induced copies of \vec{C}_4 or \vec{P}_3 , and *singular* otherwise. By Proposition 2.3, there exists an integer ℓ such that for *every* regular 3-graph G on ℓ vertices,

$$p(\rho, G) \geq \frac{4}{9} - \frac{\epsilon}{2}. \quad (7)$$

Fix for a moment an integer m such that $|V(G_m)| \geq \ell$ and let

$$R \stackrel{\text{def}}{=} \sum_{\substack{G \in \mathcal{M}_\ell \\ G \text{ is regular}}} p(G, G_m)$$

be the probability that a randomly chosen ℓ -vertex induced subgraph of G_m is regular. Given the formula (5) with $H := \rho$ and $\hat{G} := G_m$, we get from (6)

and (7) that $(4/9 - \epsilon) \geq R \left(\frac{4}{9} - \frac{\epsilon}{2} \right)$ which implies that $1 - R \geq \epsilon$. Hence, for some absolute (not depending on m) constant $\delta > 0$, there exists a *singular* 3-graph $G \in \mathcal{M}_\ell$ such that $p(G, G_m) \geq \delta$.

Now we let m vary. Since \mathcal{M}_ℓ is finite, we can assume w.l.o.g. that this singular 3-graph G is the same for all m . And now we are going to apply the “regularization” machinery reviewed at the end of the previous section to arrive at a contradiction with singularity of G .

In Claim 2.5 we set $n := 2$, $r := 3$, and let N_1 be the resulting bound. Next, we set

$$\delta' = \frac{1}{2} \ell^{-\ell} \delta, \quad (8)$$

and apply Proposition 2.6 with $n := N_1$ and $\delta := \delta'$. Let N_0 be the resulting bound, and now we fix m such that $|V(G_m)| \geq \ell N_0$. W.l.o.g. we may assume that $|V(G_m)|$ is divisible by ℓ , and let $N \stackrel{\text{def}}{=} \frac{1}{\ell} |V(G_m)|$. Note for the record that $N \geq N_0$.

Let $V(G) = [\ell]$. Consider a random balanced partition $V(G_m) = \mathbf{B}_1 \dot{\cup} \dots \dot{\cup} \mathbf{B}_\ell$ into N -sets. By a standard averaging argument, the expectation of the density of induced embeddings $\alpha : G \rightarrow G_m$ such that $\alpha(i) \in \mathbf{B}_i$ ($i \in [\ell]$) is at least δ' (recall that δ' is given by (8)). Fix an arbitrary balanced partition $V(G_m) = B_1 \dot{\cup} \dots \dot{\cup} B_\ell$ with this property, and let $S \subseteq [B_1] \times \dots \times [B_\ell]$ consist of those tuples (v_1, \dots, v_ℓ) for which the mapping $\alpha : [\ell] \rightarrow V(G_m)$ given by $\alpha(i) = v_i$ does define an induced embedding of G .

Applying Proposition 2.6, we find $A_i \subseteq B_i$ with $|A_i| = N_1$ and $A_1 \times \dots \times A_\ell \subseteq S$. And applying Claim 2.5 (with $B_i := A_i$) to the 2-coloring of $[A_1 \cup \dots \cup A_\ell]^3$ defined by the set of edges of G_m , we find distinct pairs of elements $a_i, b_i \in A_i$ such that for any $i \neq j \in [\ell]$,

$$(a_i b_i a_j) \in E(G_m) \equiv (a_i b_i b_j) \in E(G_m).$$

Note also for the record that for any $1 \leq i < j < k \leq \ell$ and for any $c_i \in \{a_i, b_i\}$, $c_j \in \{a_j, b_j\}$, $c_k \in \{a_k, b_k\}$, $(c_i c_j c_k) \in E(G_m)$ if and only if $(ijk) \in E(G)$, for any choice of the representatives c_i, c_j, c_k .

We define the directed graph Γ on $[\ell]$ by introducing a directed edge $\langle i, j \rangle$ if and only if $(a_i b_i a_j), (a_i b_i b_j) \notin E(G_m)$ (note the negation!) As the first observation, since $\{a_i, b_i, a_j, b_j\}$ span at least one 3-edge, $\langle i, j \rangle$ and $\langle j, i \rangle$ can not simultaneously belong to $E(\Gamma)$, therefore Γ is actually an oriented graph.

Now we claim that $E(G) \supseteq E(FDF(\Gamma))$ and that Γ does not contain induced copies of \bar{P}_3 and \vec{C}_4 ; this will contradict the singularity of G . In the

case analysis below (that we split into a sequence of simple claims), i, j, k stand for arbitrary pairwise different elements of $[\ell]$.

Claim 3.1 *If $\langle i, j \rangle \in E(\Gamma)$ and $\langle i, k \rangle \in E(\Gamma)$ then $(ijk) \in E(G)$.*

Proof. Since $\{a_i, b_i, a_j, a_k\}$ is not independent in G_m , at least one of the two triples $(a_i a_j a_k)$, $(b_i b_j b_k)$ must be in $E(G_m)$ and this implies $(ijk) \in E(G)$. ■

The remaining analysis does require the assumption that the 3-graphs H_1, H_2, H_3 are missing.

Claim 3.2 *If i, j are independent in Γ , and i, k are also independent in Γ , then $(ijk) \in E(G)$.*

Proof. Note first that the assumption of our claim simply says that both sets $\{a_i, b_i, a_j, b_j\}$ and $\{a_i, b_i, a_k, b_k\}$ span a clique in G_m . By symmetry, we can assume w.l.o.g. that $\langle j, k \rangle \notin E(\Gamma)$, that is, $(a_j b_j a_k), (a_j b_j b_k) \in E(G_m)$. But then since $\{a_i, b_i, a_j, b_j, a_k\}$ does not span a copy of H_1 , at least one of the four edges $(c_i c_j a_k)$ ($c_i \in \{a_i, b_i\}$, $c_j \in \{a_j, b_j\}$) must be present in G_m which implies $(ijk) \in E(G)$. ■

Reviewing the definition of $FDF(\Gamma)$, we see that we have proved $E(FDF(\Gamma)) \subseteq E(G)$. We still have to verify that Γ is \bar{P}_3 -free and \vec{C}_4 -free.

Claim 3.3 *If i, j are independent in Γ and $\langle j, k \rangle \in E(\Gamma)$ then $(ijk) \notin E(G)$.*

Proof. We again look at the configuration spanned by $\{a_i, b_i, a_j, b_j, a_k\}$. Assuming $(ijk) \in E(G)$, the only 3-edges that are missing here are $(a_j b_j a_k)$ and, possibly, $(a_i b_i a_k)$. Which gives us either H_2 or H_3 , and this contradiction proves that $(ijk) \notin E(G)$. ■

Now, Γ may not contain an induced copy of \bar{P}_3 : the case when the three vertices spanning \bar{P}_3 in Γ do not form an edge of G is taken care of by Claim 3.2, and the case when it is an edge is ruled out by Claim 3.3. Also, Γ may not contain a copy of \vec{C}_4 : since these four vertices may not be independent in G , three of them must form an edge which is again in contradiction with Claim 3.3.

We have arrived at the desired contradiction by constructing a Fon-der-Flaass realization Γ for a spanning subgraph of the 3-graph G . This completes the proof of our main result.

4. Miscellaneous calculations

We include here two other results of a similar flavor, albeit seemingly less interesting: the additional forbidden 3-graphs are present in $FDF(\Gamma_K)$, and, in the second case, are present even in Turán’s original example.

Our first statement improves upon (1).

Theorem 4.1 $\pi_{\min}(I_4^3, H_4) = 4/9$, where H_4 is a 3-graph on $[5]$ with $E(H_4) = \{(123)(124)(234)(135)\}$.

Another viable strategy in approaching difficult extremal problems (see e.g. [FPS03]) is to try to find in a hypothetical counterexample increasingly large pieces that “correspond” to known extremal configurations. For Turán’s (3,4)-problem we have been able to verify the first step on this road.

Theorem 4.2 $\pi_{\min}(I_4^3, M_2) \geq 0.4557$, where $M_2 \stackrel{\text{def}}{=} FDF(\Gamma_K)|_{\mathbb{Z}_3 \times \{1,2\}}$ is the only extremal configuration on 6 vertices.

Since these days an interested reader can check statements like this using the publicly available Flagmatic software [FRV12], we do not present here (tedious!) results of our calculations.

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